On the Capacity of Asynchronous CDMA Systems

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***Abstract*— The total capacity per chip constrained to a given chip pulse waveform of asynchronous code division multiple access (CDMA) channels with random spreading and subject to frequency-flat fading is investigated in the large system limit. The analysis in terms of signal to interference and noise ratio (SINR) is extended to CDMA systems with linear minimum mean square error (MMSE) detectors. The system behaviour is completely described by a positive function $\upsilon(N_0, f)$ of the power spectral noise N_0 and the frequency f and both total capacity per chip and SINR of a linear MMSE detector can be expressed in terms of $\upsilon(N_0, f)$. A simple relation between the total capacities per chip of asynchronous CDMA systems with modulation based on sinc pulse waveforms and of synchronous CDMA systems is derived.**

I. Introduction

The fundamental limits of synchronous CDMA systems have been thoroughly studied in [1], [2], [3] by modelling the spreading sequences by random sequences. However, this analysis is mainly focused on synchronous CDMA systems and the assumption of synchronism is not realistic for the received signal on the uplink of a CDMA system. Therefore, it is of theoretical and practical interest to extend the analysis of CDMA systems with random spreading to the asynchronous case.

The analysis of asynchronous CDMA systems limited to symbol asynchronous but chip synchronous signals, i.e. signals whose time delays are multiple of the chip interval, is in [4], [5]. In [4] the performance of a linear MMSE detector with infinite observation window is proven to be asymptotically equivalent to the performance of a synchronous CDMA system. The performance degradation of linear detectors with finite observation windows has been analyzed in [5]. In [6] linear MMSE detectors for asynchronous CDMA systems with modulation based on an ideal Nyquist sinc function (bandwidth equal to half of the chip rate $\frac{1}{T_c}$) are shown to be equivalent in terms of performance to linear MMSE detectors for synchronous CDMA systems. The effects of chip asynchronism on the performance of linear multistage detectors have been object of study in [7], [8]. Asynchronous CDMA systems with multistage detectors at the receiver and modulation based on chip pulse waveforms with bandwidth $B \leq \frac{1}{2T_c}$ have the same asymptotic performance as the correspondent synchronous systems. Furthermore, the performance is independent of the time delay distribution. Increasing the bandwidth of the chip waveform beyond $\frac{1}{2T_c}$, the system performance changes substantially. It depends on the time delay distribution and the equivalence between synchronous and asynchronous systems does not hold [7]. The impact of pulse shaping and the performance loss of linear multistage detectors due to the use of suboptimal statistics for asynchronous CDMA systems is in [8].

From a technical point of view, the large system performance analysis in [7], [8] is based on recursive expressions of the limit eigenvalue moments of the system covariance matrix $\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^H$ (here $\boldsymbol{\mathcal{H}}$ is the transfer matrix of the system) and not on the whole limit eigenvalue distribution required to derive the total capacity per chip of a large CDMA system or its SINR at the output of a linear MMSE detector (see [1], [2], [3]). Thus, the results in [7], [8] do not enable a large system analysis of the effects of asynchronism on these two relevant performance measures. In this work we investigate the fundamental limits of asynchronous CDMA systems in terms of both total capacity per chip constrained to a chip pulse waveform and the limit SINR of linear MMSE detectors.

As already shown in [7] for CDMA systems with linear multistage detectors, synchronous and asynchronous CDMA systems are equivalent in terms of total capacity per chip and SINR at the output of a linear MMSE detector if the bandwidth of the chip waveform satisfies $B \leq \frac{1}{2T_c}$. In contrast, when $B > \frac{1}{2T_c}$ asynchronous CDMA systems outperform synchronous systems.

Due to space restriction in this work we consider (i) CDMA systems with modulation based on chip pulse waveforms with bandwidth $B \leq \frac{1}{2T_c}$ and any set of time delays or (ii) CDMA systems using chip pulse waveforms with bandwidth $B > \frac{1}{2T_c}$ and uniform the time delay distribution. For the general case of CDMA with any time delay distribution the interested reader can refer to [9]. In both case (i) and (ii) the behaviour of large CDMA systems with linear MMSE detectors is completely described by a positive scalar function $\upsilon(N_0, f)$ of the power spectral noise N_0 and the frequency f.

In [2] it was shown that in large synchronous CDMA systems using random spreading sequences, the limiting interference effects under linear MMSE detection can be decoupled. The level of interference that can be ascribed to an interferer k is referred to as effective interference. Beside the decoupling effects on interferers as in synchronous CDMA systems the large system analysis of asynchronous systems shows an additional decoupling effect in frequency so that the concept of spectrum of the effective interference can be introduced. The effective interference at frequency f of a user k on the user of interest is the level of interference that can be attributed to the component of the signal of user k at frequency f in the detection of the user of interest.

Furthermore, in a large asynchronous CDMA system with number of transmitted symbols per chip β and modulation based on a sinc function with bandwidth $B = \frac{\gamma}{2T_c}$ we show that a linear MMSE detector performs as well as in a synchronous system with modulation based on square root Nyquist pulses and system load $\beta' = \frac{\beta}{\gamma}$. This property implies the possibility to trade degrees of freedom in the frequency domain provided by the bandwidth of the chip pulse waveform against the degrees of freedom in the time domain provided by the spreading factor N.

For CDMA systems (i) and (ii) also the total capacity per chip can be expressed as a function of $v(N_0, f)$. An explicit expression for the constrained total capacity is provided for modulation based on a sinc function with bandwidth $B = \frac{\gamma}{2T_c}$. In this case the constrained total capacity per chip of a system with load β is γ times the total capacity per chip of a synchronous system with system load $\beta' = \frac{\beta}{\gamma}$. In synchronous CDMA systems the maximum total capacity per chip constrained to a given bandwidth $\frac{\gamma}{2T_c}$ is achieved by modulation based on square root Nyquist functions and it is constant for any $\gamma \in [1, +\infty]$. In contrast, the capacity constrained to a sinc pulse increases with the bandwidth in asynchronous CDMA systems and asynchronous systems outperform synchronous ones. The gap between the spectral efficiency of a (synchronous or asynchronous) CDMA system with ideal Nyquist sinc pulse ($\gamma = 1$) and the spectral efficiency of an asynchronous system using a sinc pulse with any $\gamma > 1$ vanishes asymptotically as $\beta \to \infty$ if the energy per bit per noise level $\frac{E_b}{N_0}$ is kept constant.

Due to space restriction the proofs of the theorems are omitted in this work. The interested reader can refer to [9].

II. System Model

Let us consider an asynchronous CDMA system with K users and spreading factor N in an uplink fading channel impaired by additive white Gaussian noise (AWGN). Then, $\beta = \frac{K}{N}$ is the system load and the signal received at the base station, in complex base-band notation, is given by

$$y(t) = \sum_{k=1}^{K} a_{kk} s_k(t - \tau_k) + n(t) \qquad t \in [-\infty, +\infty].$$

Here, a_{kk} is the received signal amplitude of user k; τ_k is the time delay of user k; $n(t)$ is a zero mean white, complex Gaussian process with two-sided power spectral density N_0; and $s_k(t)$ is the spread signal of user k. We have

$$s_k(t) = \sum_{m=-\infty}^{+\infty} b_k[m] c_k^{(m)}(t),$$

where $b_k[m]$ is the m^{th} transmitted symbol of user k and

$$c_k^{(m)}(t) = \sum_{u=0}^{N-1} s_{k,m}[u] \psi(t - mT_s - uT_c)$$

is its spreading waveform at time m. Here, $\boldsymbol{s}_{k,m}$, is the spreading sequence of user k in the m^{th} symbol interval with elements $s_{k,m}[u]$, $u = 0, \ldots, N-1$; T_s and $T_c = \frac{T_s}{N}$ are the symbol and chip periods, respectively.

The users' symbols $b_k[m]$ are uncorrelated and identically distributed random variables with $\mathrm{E}\{|b_k[m]|^2\} = 1$ and $\mathrm{E}\{b_k[m]\} = 0$. The elements of the spreading sequences $s_{k,m}[u]$ are assumed to be i.i.d. random variables with $\mathrm{E}\{|s_{k,m}[u]|^2\} = \frac{1}{N}$ and $\mathrm{E}\{s_{k,m}[u]\} = 0$.

The chip waveform $\phi(t)$ is bandlimited with bandwidth B, unit energy, and Fourier transform $\Phi(f)$. Thanks to the statistical properties of the spreading sequences, the average energy of the signature waveform is also unit.

At the front-end the base band signal is processed by a lowpass filter with lowpass band $B_{\text{FE}} = \frac{r}{2T_c}$ and $r \geq 2BT_c$. Then, the chip pulse waveform at the output of the low pass filter is still $\Phi(f)$. The filter output is sampled at rate $\frac{r}{T_c}$ such that the conditions of the sampling theorem are satisfied. With this choice of the front-end the sampled signal provides sufficient statistics and the discrete-time noise is still white with zero mean and variance $\sigma^2 = \frac{N_0 r}{T_c}$.

The discrete-time signal at the front-end output is given by

$$y[p] = \sum_{k=1}^{K} a_k \sum_{m=-\infty}^{+\infty} b_k[m] \sum_{u=0}^{N-1} s_{k,m}[u] \phi\Big(\frac{p}{r}T_c - \tau_k - (u + mN)T_c\Big) + n[p] \tag{1}$$

with $p \in \mathbb{Z}$ and $n[p]$ the discrete-time, complex-valued noise.

Throughout this work we assume that the filtered chip pulse waveform $\phi(t)$ is much shorter than the symbol waveform, i.e. $\phi(t)$ becomes negligible for $|t| > t_0$ and $t_0 \ll T_s$. This is usually verified in the systems with large spreading factor, which we are considering. Thus, we can neglect the intersymbol interference. Then, given the time delay τ_k the virtual spreading sequence of user k for the transmitted symbol m spans the symbol intervals m and $m+1$ and it is a $2Nr$-dimensional vector given by

$$\boldsymbol{v}_{km} = \boldsymbol{\Phi}_k \boldsymbol{s}_{km}$$

where $\boldsymbol{s}_{km} = (s_{km}[0] \ldots s_{km}[N-1])^T$ and $\boldsymbol{\Phi}_k$ is a $2Nr \times N$ matrix taking into account the effects of the pulse shape and the time delay of user k. The matrix $\boldsymbol{\Phi}_k$ is of the form

$$\boldsymbol{\Phi}_k = \begin{bmatrix} \mathbf{0}_{k,0}^T & \boldsymbol{C}_{\phi,r}^T\left(\tau_k - \lfloor \frac{\tau_k}{T_c} \rfloor T_c\right) & \mathbf{0}_{k,1}^T \end{bmatrix}^T \tag{2}$$

where $\mathbf{0}_{k,0}$ and $\mathbf{0}_{k,1}$ are matrices of dimensions $\lfloor \frac{r\tau_k}{T_c} \rfloor \times N$ and $\left(N - \lfloor \frac{r\tau_k}{T_c} \rfloor\right) \times N$, respectively, with zero elements; $\boldsymbol{C}_{\phi,r}(\tau_k)$ is an r-block-wise circulant matrix[1] of order N defined by

$$\boldsymbol{C}_{\phi,r}(\tau) \triangleq \boldsymbol{C}\left(\phi(x,\tau), \phi\left(x, \tau - \frac{T_c}{r}\right), \ldots, \phi\left(x, \tau - \frac{(r-1)T_c}{r}\right)\right), \tag{3}$$

with

$$\phi(x,\tau) \triangleq \frac{1}{T_c} \sum_{s=-\infty}^{+\infty} \mathrm{e}^{j2\pi \frac{\tau}{T_c}(x+s)} \Phi^*\left(\frac{j2\pi}{T_c}(x+s)\right). \tag{4}$$

The zero matrices take into account the fact that we neglect the the useful signal outside the interval $[mT_s + \tau_k, (m+1)T_s + \tau_k]$. For K and N finite, the circulant matrix $\boldsymbol{C}_{\phi,r}(\tau)$ as defined in (3) approximates the matrix directly derivable from (1), which is Toeplitz. Furthermore, the two matrices are

[1] An r-blockwise circulant matrix of order N is an $rN \times N$ matrix of N block rows of dimensions $r \times N$ such that each block is obtained by circularly right shifting of the previous block. It is completely characterized by the r Fourier transforms of each of the rows.

asymptotically equivalent in terms of spectral distribution (see e.g. [10]).

Let $\boldsymbol{S}[m]$ be the $2rN \times K$ matrix of virtual spreading, i.e. $\boldsymbol{S}[m] = (\boldsymbol{\Phi}_1 \boldsymbol{s}_{1m}, \boldsymbol{\Phi}_2 \boldsymbol{s}_{2m}, \ldots \boldsymbol{\Phi}_K \boldsymbol{s}_{Km})$, $\boldsymbol{A}$ the $K \times K$ diagonal matrix of received amplitudes, $\boldsymbol{H}[m] = \boldsymbol{S}[m]\boldsymbol{A}$, and $\boldsymbol{b}[m]$ and $\boldsymbol{y}[m]$ the vectors of transmitted and received signals, respectively. Furthermore, we decompose the matrix $\boldsymbol{H}[m]$ into two matrices of size $rN \times K$, $\boldsymbol{H}_u[m]$ and $\boldsymbol{H}_d[m]$ such that $\boldsymbol{H}[m] = [\boldsymbol{H}_u^T[m], \boldsymbol{H}_d^T[m]]^T$. Then, the baseband discrete-time asynchronous system in matrix notation is given by

$$\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{N}} \tag{5}$$

where $\boldsymbol{\mathcal{Y}} = [\ldots, \boldsymbol{y}^T[m-1], \boldsymbol{y}^T[m], \boldsymbol{y}^T[m+1] \ldots]^T$ and $\boldsymbol{\mathcal{B}} = [\ldots, \boldsymbol{b}^T[m-1], \boldsymbol{b}^T[m], \boldsymbol{b}^T[m+1] \ldots]^T$ are the infinite-length vectors of received and transmitted symbols respectively; $\boldsymbol{\mathcal{N}}$ is an infinite-length noise vector; and $\boldsymbol{\mathcal{H}}$ is a bi-diagonal block matrix with infinite block rows and block columns given by

$$\boldsymbol{\mathcal{H}} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ldots & \mathbf{0} & \boldsymbol{H}_d[m-1] & \boldsymbol{H}_u[m] & \mathbf{0} & \ldots & \ldots \\ \ldots & \ldots & \mathbf{0} & \boldsymbol{H}_d[m] & \boldsymbol{H}_u[m+1] & \mathbf{0} & \ldots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{6}$$

Additionally, $\boldsymbol{h}_{k,m}$ denotes the column of the matrix $\boldsymbol{\mathcal{H}}$ containing the k^{th} column of the matrix $\boldsymbol{H}[m]$.

III. Linear MMSE Detection

The linear MMSE detector $\boldsymbol{\mathcal{C}}_{k,m}$ generates a soft decision $\widehat{b}_k[m] = \boldsymbol{\mathcal{C}}_{k,m}^H \boldsymbol{\mathcal{Y}}$ of the transmitted symbol $b_k[m]$ based on the observation $\boldsymbol{\mathcal{Y}}$. It is given by

$$\boldsymbol{\mathcal{C}}_{k,m} = (\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^H + \sigma^2 \boldsymbol{I})^{-1} \boldsymbol{h}_{k,m}. \tag{7}$$

The $\mathrm{SINR}_{k,m}$ at its output of the linear MMSE detector is given by

$$\mathrm{SINR}_{k,m} = \boldsymbol{h}_{k,m}^H (\boldsymbol{\mathcal{H}}_{k,m} \boldsymbol{\mathcal{H}}_{k,m}^H + \sigma^2 \boldsymbol{I})^{-1} \boldsymbol{h}_{k,m} \tag{8}$$

where, $\boldsymbol{\mathcal{H}}_{k,m}$ is the matrix obtained from $\boldsymbol{\mathcal{H}}$ suppressing the column $\boldsymbol{h}_{k,m}$.

Deeper insight on the linear MMSE behaviour is obtained by analyzing the performance, as $K, N \to \infty$ with constant ratio β. In order to determine the asymptotic SINR at the output of a linear MMSE detector we focus on CDMA systems such that the time delays of the received signals τ_k, $k = 1, \ldots, K$ are not greater than the chip delay T_c, i.e. $\tau_k \leq T_c$, $k = 1, \ldots, K$. We refer to them as *symbol quasi-synchronous but chip asynchronous* CDMA systems. Then, $\boldsymbol{\Phi}_k = \left[\begin{array}{cc} \boldsymbol{C}_{\phi,r}^T(\tau_k) & \mathbf{0}_N^T \end{array} \right]^T$, being $\mathbf{0}_N$ an $N \times N$ zero matrix. The matrix $\boldsymbol{\mathcal{H}}$ in (6) reduce to a block diagonal matrix with blocks of dimensions $rN \times K$ and we can focus on the transmission in a single symbol interval. The virtual spreading matrix in the m^{th} symbol interval is given by

$$\overline{\boldsymbol{S}}[m] = [\boldsymbol{C}_{\phi,r}(\tau_1)\boldsymbol{s}_{1m}, \boldsymbol{C}_{\phi,r}(\tau_2)\boldsymbol{s}_{2m} \ldots \boldsymbol{C}_{\phi,r}(\tau_K)\boldsymbol{s}_{Km}]$$

Let $\overline{\boldsymbol{H}}[m] = \overline{\boldsymbol{S}}[m]\boldsymbol{A}$ be the transfer matrix of the system at time instant m^{th}. Without ambiguity we can drop the index m in the following and denote and let $\overline{\boldsymbol{R}} = \overline{\boldsymbol{H}}^H \overline{\boldsymbol{H}}$.

The following theorem provides the limit SINR at the output of a linear MMSE detector.

Theorem 1 *Let $\boldsymbol{A} \in \mathbb{C}^{K \times K}$ be a diagonal matrix with k^{th} diagonal element a_{kk} and T_c a positive real. Given a function $\Phi(j2\pi f) : \mathbb{R} \to \mathbb{C}$, let $\phi(x, \tau)$ be as in (4). Given $\{\widetilde{\tau}_1, \widetilde{\tau}_2 \ldots \widetilde{\tau}_K\}$ a set of reals in $[0, T_c]$ and an integer positive r, $\boldsymbol{C}_{\phi,r}(\widetilde{\tau}_k)$, $k = 1, \ldots, K$, are K r-block-wise circulant matrices of order N defined in (3). Let $\overline{\boldsymbol{H}} = \overline{\boldsymbol{S}}\boldsymbol{A}$ with $\overline{\boldsymbol{S}} = (\boldsymbol{C}_{\phi,r}(\widetilde{\tau}_1)\boldsymbol{s}_1, \boldsymbol{C}_{\phi,r}(\widetilde{\tau}_2)\boldsymbol{s}_2, \ldots, \boldsymbol{C}_{\phi,r}(\widetilde{\tau}_K)\boldsymbol{s}_K)$ and $\boldsymbol{s}_k$ N-dimensional column vectors.*

We assume that the function $\Phi(j2\pi f)$ is bounded in absolute value, bandlimited with bandwidth B and $r \geq 2BT_c$. The vectors $\boldsymbol{s}_k$ are independent with i.i.d. circulant symmetric Gaussian elements $s_{nk} \in \mathbb{C}$. The elements a_{kk} of the matrix $\boldsymbol{A}$ are uniformly bounded for any K. Furthermore, one of the following sets of conditions is satisfied.

Set of conditions A: *The sequence of the empirical joint distributions $F_{|\boldsymbol{A}|^2,T}^{(K)}(\lambda, \tau) = \frac{1}{K}\sum_{k=1}^K 1(\lambda - |a_{kk}|^2) 1(\tau - \widetilde{\tau}_k)$ converges almost surely, as $K \to \infty$, to a non-random distribution function $F_{|\boldsymbol{A}|^2,T}(\lambda, \tau)$ with λ and τ statistically independent and τ uniformly distributed in $[0, T_c]$.*

Set of conditions B: *The bandwidth B satisfies the constraint $B \leq \frac{1}{2T_c}$ and the sequence of the empirical distributions $F_{|\boldsymbol{A}|^2}(\lambda) = \frac{1}{K}\sum_{k=1}^K 1(\lambda - |a_{kk}|^2)$ converges in law almost surely to a deterministic distribution function.*

Then, given the received power $|a_{kk}|^2$ and the power spectral density of the white noise N_0, the SINR of user k at the output of a linear MMSE detector for a CDMA system with transfer matrix $\overline{\boldsymbol{H}}$ converges in probability as $K, N \to \infty$ with $\frac{K}{N} \to \beta$ and r fixed to

$$\lim_{K=\beta N \to \infty} \mathrm{SINR}_k = |a_{kk}|^2 \int_{-B}^{B} |\Phi(j2\pi f)|^2 \upsilon(N_0, f) \mathrm{d}f \tag{9}$$

where $\upsilon(N_0, f)$ is the unique positive solution of the fixed point equation

$$\upsilon^{-1}(N_0, f) = N_0 + \beta |\Phi(j2\pi f)|^2 \int \frac{\lambda \mathrm{d}F_{|\boldsymbol{A}|^2}(\lambda)}{1 + \lambda \int_{-B}^{B} \upsilon(N_0, f) |\Phi(j2\pi f)|^2 \mathrm{d}f}$$

$$\text{for} \quad |f| \leq \frac{r}{2T_c}. \tag{10}$$

Under the conditions of Theorem 1 the multiuser efficiency of the linear MMSE detector for user k is independent of the user's time delay and it is the same for all users

$$\eta = N_0 \int_{-B}^{B} |\Phi(j2\pi f)|^2 \upsilon(N_0, f) \mathrm{d}f. \tag{11}$$

Interestingly, (9) and (10) admit the following interpretation. Let $P(f, \lambda) = \lambda |\Phi(j2\pi f)|^2$ be the spectrum of energy of the sampled received signal for a user having received energy λ. Let α_k be the limit SINR_k as $K, N \to \infty$ with constant ratio. Then, the result in Theorem 1 can be expressed as

$$\alpha_k = \int_{-B}^{B} \alpha_k(f) \mathrm{d}f$$

where $\alpha_k(f)$ is the limit spectrum of the SINR of user k

$$\alpha_k(f) = \frac{P(f, |a_{kk}|^2)}{N_0 + \beta \mathrm{E}_{|\boldsymbol{A}|^2}\{I(P(f, |a_{kk}|^2), P(f, \lambda), \alpha_k)\}}$$

and

$$I(P(f, |a_{kk}|^2), P(f, \lambda), \alpha_k) = \frac{P(f, |a_{kk}|^2) P(f, \lambda)}{P(f, |a_{kk}|^2) + P(f, \lambda)\alpha_k}.$$

Heuristically, this means that for large systems the spectrum of the SINR is deterministic and given by

$$\alpha_k(f) \approx \frac{P(f, |a_{kk}|^2)}{N_0 + \frac{1}{N}\sum_{\substack{j=1\\ j\neq k}}^{K} I(P(f,|a_{kk}|^2), P(f,|a_{jj}|^2), \alpha_k)}.$$

Then, the interference at the frequency f can be decoupled into a sum of the background noise and an interference term from each of the users at the same frequency f. The total interference at frequency f depends only on the received power of the user of interest at the frequency f, the received power of the interfering user at the same frequency, and the attained SINR α_k. Therefore, in asynchronous systems we have a decoupling of the effects of interferers as in synchronous systems [2] and an additional decoupling in frequency.

From the set of conditions B, since no assumption is made on the time delay distribution, the large system performance is independent of the set of time delays for $B \leq \frac{1}{2T_c}$ and synchronous and asynchronous systems have the same performance. For $B > \frac{1}{2T_c}$ the equivalence between synchronous and asynchronous systems does not hold and the large system performance does depend on the time delay distribution. A general expression that holds for any time delay distribution with support $[0, T_c]$ is omitted here due to space restriction and can be found in [9]. For general time delays $\tau_k \in [0, T_s]$ we conjecture the equivalence in performance between the asynchronous systems and a chip asynchronous but symbol quasi synchronous system with time delays $\widetilde{\tau}_k = \tau_k - \left\lfloor \frac{\tau_k}{T_c} \right\rfloor T_c$. The rationale behind this conjecture is in [9].

The sinc functions with bandwidth $B = \frac{\gamma}{2T_c}$ have a particular theoretical interest. In the following we specialize Theorem 1 to this case.

Given a positive real γ,

$$\Phi(j2\pi f) = \begin{cases} \sqrt{\frac{T_c}{\gamma}} & \text{for } |f| \leq \frac{\gamma}{2T_c}, \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

corresponds to a sinc waveform with bandwidth $B = \frac{\gamma}{2T_c}$ and unit energy. The large system SINR of user k at the output of a linear MMSE detector is given by $\alpha_k^{(\text{sinc})} = |a_{kk}|^2 \upsilon$ where υ is the unique positive solution to the fixed point equation

$$\upsilon^{-1} = N_0 + \frac{\beta}{\gamma}\int \frac{\lambda \mathrm{d}F_{|\boldsymbol{A}|^2}(\lambda)}{1+\lambda\upsilon}. \tag{13}$$

We recall that the asymptotic SINR of user k at the output of a linear MMSE detector for a synchronous CDMA system is given by [2] $\alpha_k^{(\text{syn})} = |a_{kk}|^2 \zeta$ being ζ the unique positive solution to the fixed point equation

$$\zeta^{-1} = N_0 + \beta\int \frac{\lambda \mathrm{d}F_{|\boldsymbol{A}|^2}(\lambda)}{1+\lambda\zeta}. \tag{14}$$

This result holds for synchronous CDMA systems using any chip pulse waveform with bandwidth $B \geq \frac{1}{2T_c}$ and satisfying the Nyquist criterion. Then, the comparison of (13) with (14) shows the interesting effect that an asynchronous CDMA system using a sinc function with bandwidth $B = \frac{\gamma}{2T_c}$ as chip pulse waveform performs as well as a synchronous CDMA system with system load $\beta' = \frac{\beta}{\gamma}$. This implies the possibility to trade the bandwidth of the chip pulse waveform against the spreading factor. In other words, we can trade the degrees of freedom in the frequency domain provided by the bandwidth of the chip pulse waveform against the degrees of freedom in the time domain provided by the spreading. This trading is typical of the asynchronous systems described above and does not extend to synchronous systems with the same waveform as apparent from (14).

IV. Capacity per Chip Constrained to a Chip Pulse Waveform

There exists a close relation between the total capacity of a CDMA system and the multiuser efficiency of a linear MMSE detector for the same system [1], [3]. In this section we extend the results in Section III to derive the capacity per chip of an asynchronous CDMA system constrained to a chip pulse waveform, i.e. the capacity of a CDMA system for which the chip pulse waveforms for all the users and the chip intervals are identical to a given chip pulse waveform.

The total capacity per chip for large synchronous CDMA systems with random spreading in an AWGN channel is [1]

$$\mathcal{C}^{(\text{syn})}(\beta, \text{SNR}) = \beta\log_2\left(1+\text{SNR}-\frac{1}{4}\mathcal{F}(\text{SNR},\beta)\right) + \log_2\Big(1+\beta\text{SNR} - \frac{1}{4}\mathcal{F}(\text{SNR},\beta)\Big) - \frac{\log_2 e}{4}\text{SNR}\mathcal{F}(\text{SNR},\beta) \tag{15}$$

being

$$\mathcal{F}(y,z) = \left(\sqrt{y(1+\sqrt{z})^2+1} - \sqrt{y(1-\sqrt{z})^2+1}\right)^2.$$

Consistently with the normalization adopted in the system model (1), $\text{SNR} = N_0^{-1}$. The total capacity per chip of a synchronous CDMA system is equal to $\mathcal{C}^{(\text{syn})}(\beta, \text{SNR})$ for any square root Nyquist waveform.

The total capacity per chip of symbol quasi synchronous but chip asynchronous CDMA systems constrained to a given chip pulse waveform $\phi(t)$ of bandwidth B is given by

$$C^{(\text{asyn})}(K, N, \overline{\boldsymbol{H}}, \sigma^2, \phi) = \frac{1}{N}\log_2\det\left(\boldsymbol{I} + \sigma^{-2}\overline{\boldsymbol{H}}\overline{\boldsymbol{H}}^H\right).$$

The convergence of the eigenvalue distribution of the matrix $\overline{\boldsymbol{T}} = \overline{\boldsymbol{H}}\overline{\boldsymbol{H}}^H$ yields the following expression for the total capacity per chip constrained to a given chip pulse waveform as $K, N \rightarrow \infty$ with constant ratio.

Theorem 2 *Let us adopt the same definitions and assumptions as in Theorem 1. Then, as $K, N \rightarrow \infty$ with $\frac{K}{N} \rightarrow \beta$, $C^{(asyn)}(K, N, \overline{\boldsymbol{H}}, \sigma^2, \phi)$ and $C^{(asyn)}(K, N, \widehat{\boldsymbol{H}}, \sigma^2, \phi)$ the total capacities per chip constrained to the chip pulse waveform $\phi(t)$ converge to a deterministic value*

$$\mathcal{C}^{(asyn)}(\beta, \sigma^2, \psi) = \frac{r}{\ln 2}\int_0^{\left(\frac{N_0 r}{T_c}\right)^{-1}} \frac{1}{t}\left(1 - \frac{1}{t}G(t^{-1})\right)\mathrm{d}t$$

where $G(\alpha)$ is the Stieltjes transform[2] *of the asymptotic eigenvalue distribution of $\overline{\boldsymbol{H}}\overline{\boldsymbol{H}}^H$ given by*

$$G(\alpha) = \frac{T_c^2}{r^2}\int_{-\frac{r}{2T_c}}^{\frac{r}{2T_c}} \upsilon(\alpha, f)\mathrm{d}f \tag{16}$$

[2]Throughout this work the Stieltjes transform of a probability density function (p.d.f.) $F(t)$ is defined as $G(\alpha) = \int \frac{\mathrm{d}F(t)}{\alpha+t}$

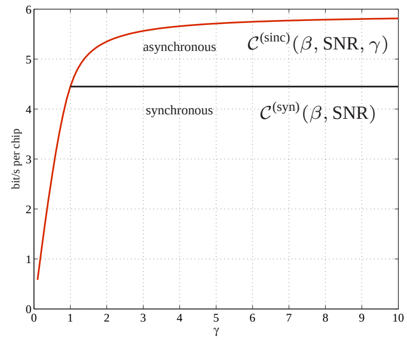

Fig. 1. Large system capacity per chip versus γ for $\beta = 1$ and $\frac{E_b}{N_0} = 10$ dB.

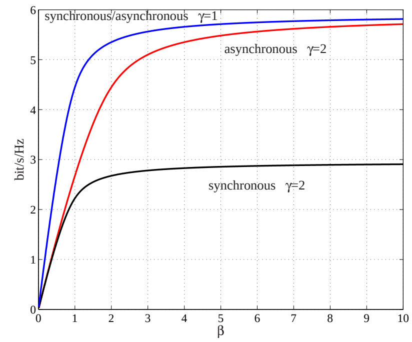

Fig. 2. Spectral efficiency versus β for $\frac{E_b}{N_0} = 10$ dB.

being $\upsilon(\alpha, x)$ the unique positive function solution to the fixed point equation in (10).

The insightful results in [3] cannot be readily extended to the analysis of asynchronous CDMA systems since they would require the multiple access interference (MAI) to be Gaussian. Nevertheless, in some special cases we can derive an explicit expression for the sum capacity constrained to a chip pulse waveform. Let us consider again the case of chip pulse waveforms defined in (12), uniform distribution of the time delays, and additive white Gaussian channel without fading. By specializing the Stieltjes transform in (16) to this case we can derive the probability density function of the eigenvalues of $\overline{\boldsymbol{T}}$

$$f(\tau) = \frac{\gamma}{r}\frac{T_c}{r}\aleph\left(\frac{T_c\tau}{r}, \frac{\beta}{\gamma}\right) + \frac{r-\gamma}{r}\delta(\tau) \qquad (17)$$

where $\aleph(y, z)$ is the deformed quarter circle law

$$\aleph(y,z) = \begin{cases} \frac{\sqrt{(y-(1-\sqrt{z})^2)((1+\sqrt{z})^2-y)}}{2\pi z y} & \text{for} \quad (1-\sqrt{z})^2 \le y \le (1+\sqrt{z})^2, \\ \max(0, 1-\beta^{-1})\delta(x) & \text{elsewhere.} \end{cases}$$

Then, using the p.d.f. in (17) and recalling that $\sigma^2 = \frac{N_0 r}{T_c}$ the total capacity per chip constrained to the chip pulse waveform (12) for large systems is

$$\mathcal{C}^{(\text{sinc})}(\beta, \text{SNR}, \gamma)\big|_{\text{SNR}=N_0^{-1}} = \gamma\, \mathcal{C}^{(\text{syn})}\left(\frac{\beta}{\gamma}, \text{SNR}\right)\Bigg|_{\text{SNR}=N_0^{-1}}. \qquad (18)$$

It is apparent from (18) that synchronous and asynchronous systems have the same capacity for $\gamma = 1$.

In order to compare different systems (with possibly different spreading gains and data rates) the total capacity per chip has to be given as a function of $\frac{E_b}{N_0}$, the level of energy per bit per noise level equal to [1] [3] $\frac{E_b}{N_0} = \frac{\beta \text{SNR}}{\mathcal{C}^{(*)}(\beta, \text{SNR}, \cdot)}$.

In Figure 1 we compare the capacities per chip $\mathcal{C}^{(\text{sinc})}(\beta, \text{SNR}, \gamma)$ for asynchronous CDMA systems with $\mathcal{C}^{(\text{syn})}(\beta, \text{SNR})$ for synchronous CDMA systems. The capacities are plotted as functions of γ with $\frac{E_b}{N_0} = 10\,\text{dB}$ and $\beta = 1$. We see that asynchronous CDMA systems outperform synchronous systems and they compensate to some extent for the loss in spectral efficiency due to the increase in bandwidth of synchronous CDMA systems. In Figure 2 we compare the spectral efficiencies $\frac{\mathcal{C}^{(*)}(\beta, \text{SNR}, \cdot)}{\gamma}$ of synchronous and asynchronous systems using the chip pulse waveform (12) for $\gamma = 1$ and $\gamma = 2$. The spectral efficiency is plotted as a function of the system load β for $\frac{E_b}{N_0} = 10$ dB. In contrast to the synchronous case, asymptotically for $\beta \to \infty$ the gap in spectral efficiency between synchronous/asynchronous systems with $\gamma = 1$ and asynchronous systems with $\gamma = 2$ vanishes.

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